arXiv:1204.0272v2 [math.CA] 3 Apr 2012

HERMITE-HADAMARD'S INEQUALITIES FOR PREINVEX FUNCTIONS VIA FRACTIONAL INTEGRALS AND RELATED FRACTIONAL INEQUALITIES

İMDAT İŞCAN

Abstract. In this paper, first we have established Hermite- Hadamard's inequalities for preinvex functions via fractional integrals. Second we extend some estimates of the right side of a Hermite- Hadamard type inequality for preinvex functions via fractional integrals.

1. Introduction and Preliminaries

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This doubly inequality is known in the literature as Hermite-Hadamard integral inequality for convex mapping.We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality . For several recent results concerning the inequality (1.1) we refer the interested reader to [1, 2, 3, 4, 5] and the references cited therein.

Definition 1. *The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the following inequality holds:*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

In [5] Pearce and Pečarić established the following result connected with teh right part of (1.1).

Theorem 1. *Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$, and let $q \geq 1$. If the mapping $|f'|^q$ convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{\frac{1}{q}} \tag{1.2}$$

Date: March 24, 2012.
2000 *Mathematics Subject Classification.* 26D10, 26D15, 26A51.
Key words and phrases. Hermite-Hadamard inequalities,invex set, preinvex function, fractional integral.

The classical Hermite- Hadamard inequality provides estimates of the mean value of a continuous convex function $f:[a,b]\rightarrow\mathbb{R}$.

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 2. *Let $f\in L[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of oder $\alpha>0$ with $a\geq 0$ are defined by*

$$J_{a^{+}}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int\limits_{a}^{x}(x-t)^{\alpha-1}f(t)dt,\ \ x>a$$

and

$$J_{b^{-}}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int\limits_{x}^{b}(t-x)^{\alpha-1}f(t)dt,\ \ x<b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function and $J_{a^{+}}^{0}f(x)=J_{b^{-}}^{0}f(x)=f(x)$.

In the case of $\alpha=1$, the fractional integral reduces to the classical integral. Properties concerning this operator can be found ([7]-[9]).

For some recent result connected with fractional integral see ([10]-[13]).

In [10] Sarıkaya et al. proved the following Hadamard type inequalities for fractional integrals as follows.

Theorem 2. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(x)+J_{b^{-}}^{\alpha}f(x)\right]\leq\frac{f(a)+f(b)}{2} \tag{1.3}$$

with $\alpha>0$.

Using the following identity Saıkaya et al. in [10] established the following result which hold for convex functions.

Lemma 1. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(x)+J_{b^{-}}^{\alpha}f(x)\right]=\frac{b-a}{2}\int\limits_{0}^{1}\left[(1-t)^{\alpha}-t^{\alpha}\right]f'(ta+(1-t)b)\,dt \tag{1.4}$$

Theorem 3. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is a convex function on $[a,b]$, then the following inequalities for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(x)+J_{b^{-}}^{\alpha}f(x)\right]\right|\leq\frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[|f'(a)|+|f'(b)|\right] \tag{1.5}$$

In recent years several extentions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [14]. Weir and Mond [15] introduced the concept of preinvex functions and applied it to the establisment of the sufficient optimality

conditions and duality in nonlinear programming. Pini [16] introduced the concept of prequasiinvex as a generalization of invex functions. Later, Mohan and Neogy [24] obtained some properties of generalized preinvex functions. Noor [17]-[19] has established some Hermite-Hadamard type inequalities for preinvex and log-preinvex functions.In recent papers Barani et al. in [21] presented some estimates of the right hand side of a Hermite-Hadamard type inequality in which some preinvex functions are involved.

In this paper we generalized the results in [21] and [10] for preinvex functions via fractional integrals. Now we recall some notions in invexity analysis which will be used throught the paper (see [22, 23] and references therein)

Let $f : A\to \mathbb{R}$ and $\eta : A\times A \to \mathbb{R}$,where A is a nonempty set in $\mathbb{R}^n$, be continuous functions.

Definition 3. *The set $A \subseteq \mathbb{R}^n$ is said to be invex with respect to $\eta(.,.)$, if for every $x, y \in A$ and $t \in [0,1]$,*

$$x + t\eta(y, x) \in A.$$

The invex set A is also called a $\eta-$connected set.

It is obvious that every convex set is invex with respect to $\eta(y, x) = y - x$, but there exist invex sets which are not convex [22].

Definition 4. *The function f on the invex set A is said to be preinvex with respect to η if*

$$f\left(x + t\eta(y, x)\right) \leq (1 - t)\, f(x) + tf(y),\ \forall x, y \in A,\ t \in [0,1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

Mohan and Neogy [24] introduced condition C defined as follows

Condition C: Let $A \subseteq \mathbb{R}^n$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. We say that the function η satisfies the condition C if for any $x, y \in A$ and any $t \in [0,1]$,

$$\begin{aligned} \eta\left(y, y + t\eta(x, y)\right) &= -t\eta(x, y) \\ \eta\left(x, y + t\eta(x, y)\right) &= (1 - t)\eta(x, y). \end{aligned}$$

Note that for every $x, y \in A$ and every $t \in [0,1]$ from condition C, we have

$$\eta\left(y + t_2\eta(x, y), y + t_1\eta(x, y)\right) = (t_2 - t_1)\eta(x, y). \tag{1.6}$$

We will use the condition in our main results.

In [20] Noor proved the Hermite-Hadamard inequality for the preinvex functions as follows:

Theorem 4. *Let $f : K = [a, a + \eta(b, a)] \to (0,\infty)$ be a preinvex function on the interval of real numbers K^o (the interior of K) and $a, b \in K^o$ with $a < a + \eta(b, a)$. Then the following inequality holds:*

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a) + f(b)}{2} \tag{1.7}$$

In [21] Barani, Gahazanfari, and Dragomir proved the following theorems:

Theorem 5. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $|f'|$ is preinvex on A then, for every $a, b \in A$ with $\eta(b,a) \neq 0$ the following inequalities holds*

$$\begin{aligned} &\left| \frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ &\leq \frac{|\eta(b,a)|}{8} \left[|f'(a)| + |f'(b)| \right]. \end{aligned} \tag{1.8}$$

Theorem 6. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. Assume that $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is preinvex on A then, for every $a, b \in A$ with $\eta(b,a) \neq 0$ the following inequalities holds*

$$\begin{aligned} &\left| \frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ &\leq \frac{|\eta(b,a)|}{2(p+1)^{\frac{1}{p}}} \left[|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \right]^{\frac{p-1}{p}}. \end{aligned} \tag{1.9}$$

2. Hermite-Hadamard type inequalities for preinvex functions via fractional integrals

Theorem 7. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b,a)$. If $f : [a, a+\eta(b,a)] \to (0, \infty)$ is a preinvex function, $f \in L[a, a+\eta(b,a)]$ and η satisfies condition C then, the following inequalities for fractional integrals holds:*

$$\begin{aligned} f\left(\frac{2a+\eta(b,a)}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)} \left[J_{a^+}^{\alpha} f(a+\eta(b,a)) + J_{(a+\eta(b,a))^-}^{\alpha} f(a) \right] \\ &\leq \frac{f(a) + f(a+\eta(b,a))}{2} \leq \frac{f(a)+f(b)}{2} \end{aligned} \tag{2.1}$$

with $\alpha > 0$.

Proof. Since $a, b \in A$ and A is an invex set with respect to η, for every $t \in [0,1]$, we have $a + t\eta(b,a) \in A$. By preinvexity of f, we have for every $x, y \in [a, a+\eta(b,a)]$ with $t = \frac{1}{2}$

$$f\left(x + \frac{\eta(y,x)}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

i.e. with $x = a + (1-t)\eta(b,a)$, $\quad y = a + t\eta(b,a)$ from inequality (1.6) we get

$$\begin{aligned} &2f\left(a + (1-t)\eta(b,a) + \frac{\eta(a+t\eta(b,a), a+(1-t)\eta(b,a))}{2}\right) \\ &= 2f\left(a + (1-t)\eta(b,a) + \frac{(2t-1)\eta(b,a))}{2}\right) = 2f\left(\frac{2a+\eta(b,a)}{2}\right) \\ &\leq f(a + (1-t)\eta(b,a)) + f(a + t\eta(b,a)) \end{aligned} \tag{2.2}$$

Multiplying both sides (2.2) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
& \frac{2}{\alpha} f\left(\frac{2a+\eta(b,a)}{2}\right) \\
\leq & \int_0^1 t^{\alpha-1} f\left(a+(1-t)\eta(b,a)\right) dt + \int_0^1 t^{\alpha-1} f\left(a+t\eta(b,a)\right) dt \\
= & \frac{1}{\eta^{\alpha}(b,a)}\left[\int_a^{a+\eta(b,a)} \left(a+\eta(b,a)-u\right)^{\alpha-1} f(u)du + \int_a^{a+\eta(b,a)} (u-a)^{\alpha-1} f(u)du\right] \\
= & \frac{\Gamma(\alpha)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+} f(a+\eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-} f(a)\right]
\end{aligned}$$

i.e.

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+} f(a+\eta(b,a)) + J^{\alpha}_{a+\eta(b,a)^-} f(a)\right]$$

and the fist inequality is proved.

For the proof of the second inequality in (2.2) we first note that if f is a preinvex function on $[a, a+\eta(b,a)]$ and the mapping η satisfies condition C then for every $t\in[0,1]$, from inequality (1.6) it yields

$$\begin{aligned}
f\left(a+t\eta(b,a)\right) &= f\left(a+\eta(b,a)+(1-t)\eta(a,a+\eta(b,a))\right) \\
&\leq tf\left(a+\eta(b,a)\right)+(1-t)f(a) \qquad (2.3)
\end{aligned}$$

and similarly

$$\begin{aligned}
f\left(a+(1-t)\eta(b,a)\right) &= f\left(a+\eta(b,a)+t\eta(a,a+\eta(b,a))\right) \\
&\leq (1-t)f\left(a+\eta(b,a)\right)+tf(a).
\end{aligned}$$

By adding these inequalities we have

$$f\left(a+t\eta(b,a)\right)+f\left(a+(1-t)\eta(b,a)\right) \leq f(a)+f\left(a+\eta(b,a)\right) \qquad (2.4)$$

Then multiplying both (2.4) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1} f\left(a+t\eta(b,a)\right) dt + \int_0^1 t^{\alpha-1} f\left(a+(1-t)\eta(b,a)\right) dt \leq \left[f(a)+f\left(a+\eta(b,a)\right)\right]\int_0^1 t^{\alpha-1}dt.$$

i.e.

$$\frac{\Gamma(\alpha)}{\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+} f(a+\eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-} f(a)\right] \leq \frac{f(a)+f\left(a+\eta(b,a)\right)}{\alpha}.$$

Using the mapping η satisfies condition C the proof is completed. □

Remark 1. *a) If in Theorem 7, we let $\eta(b,a)=b-a$, then inequality (2.1) become inequality (1.3) of Theorem 2.*

b) If in Theorem 7, we let $\alpha=1$, then inequality (2.1) become inequality (1.7) of Theorem 4.

Lemma 2. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b,a)$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If f' is preinvex function on A and $f' \in L[a, a + \eta(b,a)]$ then, the following equality holds:*

$$\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha} f(a+\eta(b,a)) + J_{(a+\eta(b,a))^-}^{\alpha} f(a)\right]$$

$$= \frac{\eta(b,a)}{2}\int_0^1 \left[t^{\alpha} - (1-t)^{\alpha}\right] f'(a + t\eta(b,a))\,dt \tag{2.5}$$

Proof. It suffices to note that

$$\begin{aligned}
I &= \int_0^1 \left[t^{\alpha} - (1-t)^{\alpha}\right] f'(a + t\eta(b,a))\,dt \\
&= \left[\int_0^1 t^{\alpha} f'(a + t\eta(b,a))\,dt\right] + \left[-\int_0^1 (1-t)^{\alpha} f'(a + t\eta(b,a))\,dt\right] \\
& \quad I_1 + I_2 \qquad (2.6)
\end{aligned}$$

integrating by parts

$$\begin{aligned}
I_1 &= \int_0^1 t^{\alpha} f'(a + t\eta(b,a))\,dt \\
&= t^{\alpha}\frac{f(a + t\eta(b,a))}{\eta(b,a)}\bigg|_0^1 - \int_0^1 \alpha t^{\alpha-1}\frac{f(a + t\eta(b,a))}{\eta(b,a)}dt \\
&= \frac{f(a+\eta(b,a))}{\eta(b,a)} - \frac{\alpha}{\eta(b,a)}\int_a^{a+\eta(b,a)}\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha-1}\frac{f(x)}{\eta(b,a)}dx \\
&= \frac{f(a+\eta(b,a))}{\eta(b,a)} - \frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(b,a)} J_{(a+\eta(b,a))^-}^{\alpha} f(a) \qquad (2.7)
\end{aligned}$$

and similarly we get,

$$\begin{aligned}
I_2 &= -\int_0^1 (1-t)^{\alpha} f'(a + t\eta(b,a))\,dt \\
&= -(1-t)^{\alpha}\frac{f(a + t\eta(b,a))}{\eta(b,a)}\bigg|_0^1 - \int_0^1 \alpha(1-t)^{\alpha-1}\frac{f(a + t\eta(b,a))}{\eta(b,a)}dt \\
&= \frac{f(a)}{\eta(b,a)} - \frac{\alpha}{\eta(b,a)}\int_a^{a+\eta(b,a)}\left(\frac{a + t\eta(b,a) - x}{\eta(b,a)}\right)^{\alpha-1}\frac{f(x)}{\eta(b,a)}dx \\
&= \frac{f(a)}{\eta(b,a)} - \frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(b,a)} J_{a^+}^{\alpha} f(a+\eta(b,a)) \qquad (2.8)
\end{aligned}$$

Using (2.7) and (2.8) in (2.6), it follows that

$$I=\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right].$$

Thus, by multiplying both sides by $\frac{\eta(b,a)}{2}$, we have conclusion (2.5). □

Remark 2. *If in Lemma 2, we let $\eta(b,a)=b-a$, then equality (2.5) become inequality (1.4) of Lemma 1.*

Theorem 8. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$ such that $f'\in L\left[a,a+\eta(b,a)\right]$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. If $|f'|$ is preinvex function on A then the following inequality for fractional integrals with $\alpha>0$ holds:*

$$\begin{aligned}(2.9)\qquad &\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right|\\ \leq\ &\frac{\eta(b,a)}{2\left(\alpha+1\right)}\left(1-\frac{1}{2^{\alpha}}\right)\left[|f'(a)|+|f'(b)|\right]\end{aligned}$$

Proof. Using lemma 2 and the preinvexity of $|f'|$ we get

$$\begin{aligned}&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right|\\
\leq\ &\frac{\eta(b,a)}{2}\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|dt\\
\leq\ &\frac{\eta(b,a)}{2}\int_{0}^{1}\left|t^{\alpha}-(1-t)^{\alpha}\right|\left[(1-t)\left|f'\left(a\right)\right|+t\left|f'\left(b\right)\right|\right]dt\\
\leq\ &\frac{\eta(b,a)}{2}\left\{\int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left[(1-t)\left|f'\left(a\right)\right|+t\left|f'\left(b\right)\right|\right]dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}-(1-t)^{\alpha}\right]\left[(1-t)\left|f'\left(a\right)\right|+t\left|f'\left(b\right)\right|\right]dt\right\}\\
=\ &\frac{\eta(b,a)}{2}\left[\left|f'\left(a\right)\right|+\left|f'\left(b\right)\right|\right]\left(\int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt\right)\\
=\ &\frac{\eta(b,a)}{2\left(\alpha+1\right)}\left(1-\frac{1}{2^{\alpha}}\right)\left[|f'(a)|+|f'(b)|\right],\end{aligned}$$

which completes the proof. □

Remark 3. *a) If in Theorem 8, we let $\eta(b,a)=b-a$, then inequality (2.9) become inequality (1.5) of Theorem 3.*

b) If in Theorem 8, we let $\alpha=1$, then inequality (2.9) become inequality (1.8) of Theorem 5.

c) In Theorem 8, assume that η satisfies condition C and using inequality (2.3) we get

$$\begin{aligned}&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\right]\right|\\ \leq\ & \frac{\eta(b,a)}{2\left(\alpha+1\right)}\left(1-\frac{1}{2^{\alpha}}\right)\left[\left|f'(a)\right|+\left|f'(a+\eta(b,a))\right|\right]\end{aligned}$$

Theorem 9. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$ such that $f'\in L\left[a,a+\eta(b,a)\right]$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. If $\left|f'\right|^{q}$ is preinvex function on A for some fixed $q>1$ then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\right]\right|\\ \leq\ & \frac{\eta(b,a)}{2\left(\alpha p+1\right)^{\frac{1}{p}}}\left(\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2}\right)^{\frac{1}{q}}\end{aligned}\tag{2.10}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha\in\left[0,1\right]$.

Proof. From lemma2 and using Hölder inequality with properties of modulus, we have

$$\begin{aligned}&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^+}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\right]\right|\\ \leq\ & \frac{\eta(b,a)}{2}\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|dt\\ \leq\ & \frac{\eta(b,a)}{2}\left(\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(a+t\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}.\end{aligned}$$

We know that for $\alpha\in[0,1]$ and $\forall t_1,t_2\in[0,1]$,

$$\left|t_1^{\alpha}-t_2^{\alpha}\right|\leq\left|t_1-t_2\right|^{\alpha},$$

therefore

$$\begin{aligned}\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|^{p}dt &\leq \int_0^1\left|1-2t\right|^{\alpha p}dt\\ &= \int_0^{\frac{1}{2}}\left[1-2t\right]^{\alpha p}dt+\int_{\frac{1}{2}}^1\left[2t-1\right]^{\alpha p}dt\\ &= \frac{1}{\alpha p+1}.\end{aligned}$$

Since $\left|f'\right|^{q}$ is convex on $\left[a,a+\eta(b,a)\right]$, we have inequality (2.10), which completes the proof. □

Remark 4. *a) If in Theorem 9, we let $\eta(b,a)=b-a$ and $\alpha=1$then inequality (2.10) become inequality (1.9) of Theorem6.*

b) In Theorem 9, assume that η satisfies condition C and using inequality (2.3) we get

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\
\leq\quad &\frac{\eta(b,a)}{2\left(\alpha p+1\right)^{\frac{1}{p}}}\left(\frac{|f'(a)|^{q}+|f'(a+\eta(b,a))|^{q}}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 10. *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$ such that $f'\in L\left[a,a+\eta(b,a)\right]$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. If $|f'|^{q}$ is preinvex function on A for some fixed $q>1$ then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\
\leq\quad &\frac{\eta(b,a)}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[\frac{|f'(a)|^{q}+|f'(b)|^{q}}{2}\right]^{\frac{1}{q}}
\end{aligned}\tag{2.11}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha>0$.

Proof. From lemma2 and using Hölder inequality with properties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J_{a^{+}}^{\alpha}f(a+\eta(b,a))+J_{(a+\eta(b,a))^{-}}^{\alpha}f(a)\right]\right| \\
\leq\quad &\frac{\eta(b,a)}{2}\int_{0}^{1}|t^{\alpha}-(1-t)^{\alpha}|^{\frac{1}{p}+\frac{1}{q}}\left|f'\left(a+t\eta(b,a)\right)\right|dt \\
\leq\quad &\frac{\eta(b,a)}{2}\left(\int_{0}^{1}|t^{\alpha}-(1-t)^{\alpha}|\,dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}|t^{\alpha}-(1-t)^{\alpha}|\left|f'\left(a+t\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_{0}^{1}|t^{\alpha}-(1-t)^{\alpha}|\,dt &= \int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}-(1-t)^{\alpha}\right]dt \\
&= \frac{2}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right).
\end{aligned}$$

Since $|f'|^{q}$ is preinvex function on A, we obtain

$$\left|f'\left(a+t\eta(b,a)\right)\right|^{q}\leq(1-t)\left|f'(a)\right|^{q}+t\left|f'(b)\right|^{q},\quad t\in[0,1]$$

and

$$\begin{aligned}
\int_0^1 |t^\alpha-(1-t)^\alpha| \left|f'\left(a+t\eta(b,a)\right)\right|^q dt &\leq \int_0^1 |t^\alpha-(1-t)^\alpha| \left[(1-t)\left|f'(a)\right|^q+t\left|f'(b)\right|^q\right] dt \\
&= \int_0^{\frac{1}{2}} \left[(1-t)^\alpha-t^\alpha\right] \left[(1-t)\left|f'(a)\right|^q+t\left|f'(b)\right|^q\right] dt \\
&\quad+\int_{\frac{1}{2}}^1 \left[t^\alpha-(1-t)^\alpha\right] \left[(1-t)\left|f'(a)\right|^q+t\left|f'(b)\right|^q\right] dt \\
&= \frac{1}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right)\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]
\end{aligned}$$

from here we obtain inequality (2.11) which completes the proof. □

Remark 5. *a) If in Theorem10, we let $\eta(b,a)=b-a$ and $\alpha=1$ then inequality (2.11)become inequality (1.2) Theorem1.*

b) In Theorem10, assume that η satisfies condition C and using inequality (2.3) we get

$$\begin{aligned}
&\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\left[J_{a^+}^\alpha f(a+\eta(b,a))+J_{(a+\eta(b,a))^-}^\alpha f(a)\right]\right| \\
\leq\quad & \frac{\eta(b,a)}{(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left[\frac{\left|f'(a)\right|^q+\left|f'(a+\eta(b,a))\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

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Department of Mathematics, Faculty of Science and Arts, Giresun University, Giresun, Turkey

E-mail address: imdat.iscan@giresun.edu.tr